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Diffusion in Layered Random Flows, Polymers, Electrons in Random Potentials, and Spin Depolarization in Random Fields

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Several related models are studied in a common framework. We first reconsider the model of Matheron and de Marsilly for (anomalous) tracer dispersion in a stratified porous medium. In each horizontal layer the flow velocity is constant, parallel to the layer, and depends randomly on the vertical coordinate z. This model is mapped onto a $d=1$ localization problem in a random potential and, equivalently, onto a $d=1$ polymer. At large t the *averaged* distribution of horizontal displacements x takes the scaling form $[P(x, t, z=0)] = at^{-5/4}Q(bxt^{-3/4})$, where $Q(y)$ is independent of the details of the model. $Q(y)$, a, and b are obtained exactly for a large class of models. From the Lifschitz tails of the localization problem we find in the region $x \gg t^{3/4}$, i.e., $y \to \infty$, that $Q(y) \sim |y| \exp(-C|y|^{4/3})$. We also obtain exactly in $d=1$ the scaling functions for the local and total average magnetization of spins diffusing in a random magnetic field, by mapping onto a polymer problem, as well as the average local concentration for diffusion in the presence of random sources and sinks. These mappings are then used to study higher-dimensional extensions of these models.

KEY WORDS: Diffusion in random flows; random potentials; spin depolarization; self-avoiding chain.

1. INTRODUCTION

In this paper we study three models of disordered systems where diffusion plays an important role: anomalous dispersion in layered random flows, diffusion with random sources and sinks, and spin depolarization in random fields. As we find, all of these models have some relation to each other and to two well-studied problems: the electron in a random potential

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and, equivalently, the Edwards self-repulsive chain. The precise connection between these last two models and its consequences in $d=1$ was described by Thouless in his beautiful work on the analytic properties of the averaged Green function of an electron in a random potential. Our aim here is to use the work of Thouless to obtain new exact results for these related models, which are quite interesting physically. That all these problems are related seems quite natural since mathematically they all amount to obtaining some information on the distribution of the random variable $\int_0^t V(z(\tau))\,dt$, where $z(\tau)$ is a Wiener process (Brownian walk) and $V(z)$ an arbitrary (random) function. This question, of course, has been much studied in probability theory and some of the references are reviewed in the Conclusion. However, despite recent renewed interest in the physics literature in the models studied here, these connections between the various models, which we find quite useful, have not been fully exploited. We thus believe that the detailed presentation of Section 2 might be useful, even if some of these connections are quite natural and, by now, probably well known to probabilists. Let us now introduce the models for which new results are obtained.

1.1. Diffusion in a Random Layered Flow

A large number of recent theoretical works[1-8] have addressed the problem of classical diffusion of independent particles in a random environment with spatial quenched disorder. Apart from the $d=1$ case (and marginally $d=2$)[1-4] or from media with "long-tailed" distributions of local disorder,[5] the most interesting anomalous diffusion behaviors have been shown to occur when the disorder of the medium has long-range correlations.[6-8] Such correlations could happen for incompressible flows in porous materials if the flow lines are correlated over large distances. A particularly simple, yet interesting model of divergenceless flow realizing this idea was introduced[9] a long time ago in hydrology to study the permeability of stratified porous media. More recently, Matheron and de Marsilly (MdM) showed[10] that this model can lead to anomalous diffusion (we thus refer to it as the MdM model). They considered the motion of a test particle in a 2D layered velocity flow described by the following Langevin equation:

$$\frac{dx}{dt} = V(z) + \eta_h(t), \qquad \frac{dz}{dt} = \eta(t) \tag{1.1}$$

where x and z are, respectively, the horizontal and vertical coordinates of the particle at time t. The velocity flow is horizontal and is only a

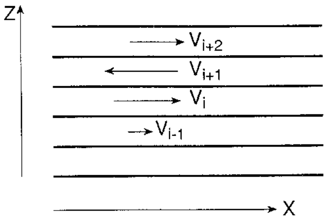


Fig. 1. Discretized version of the 2D random layered velocity flow. The velocity is along x and is a random function of the vertical coordinate z.

(random) function of the elevation z (see Fig. 1). η_h and η are two independent thermal white noise processes, i.e., with centered Gaussian distributions such that $\langle \eta(t)\, \eta(t') \rangle = 2D\delta(t-t')$ and $\langle \eta_h(t)\, \eta_h(t') \rangle = 2D_h\, \delta(t-t')$. We denote thermal averages by $\langle \cdots \rangle$ and configurational averages over the disorder of the medium [i.e., over all possible $V(z)$] by $[\cdots]$. Since the vertical motion is simply pure Brownian diffusion, (1.1) can be integrated into [with $x(0) = z(0) = 0$]:

$$x(t) = \int_0^t V(z(u))\, du + \int_0^t \eta_h(u)\, du, \qquad z(u) = \int_0^u \eta(v)\, dv \tag{1.2}$$

The horizontal displacement $x(t)$ can thus be expressed as some integral along the path of the pure Wiener process $z(u)$ $(0 \leqslant u \leqslant t)$. In the absence of bias $[V(z)] = 0$, the second moment $[\langle x(t)^2 \rangle]$ is readily evaluated[10] as

$$[\langle x(t)^2 \rangle] = 2 \int_0^t du' \int_0^{u'} du \int dz\, dz'\, [V(z)\, V(z')]\, P_0(z'-z, u'-u)\, P_0(z, u) + 2D_h t \tag{1.3}$$

We denote by $P_0(z, u) = (4\pi D u)^{-1/2} \exp(-z^2/4Du)$ the free diffusion kernel $(u > 0)$. If $[V(z)\, V(z')] = \sigma\delta(z'-z)$, (1.3) further simplifies and

$$[\langle x(t)^2 \rangle] = \int_0^t du' \int_0^{u'} du \frac{2\sigma}{[4\pi D(u'-u)]^{1/2}} + 2D_h t = \frac{4\sigma}{3(\pi D)^{1/2}}\, t^{3/2} + 2D_h t \tag{1.4}$$

{we will find that more generally (see Appendix) the leading term is the same with σ replaced by $\sigma \equiv \int dz\, [V(0)\, V(z)]$}. Thus, as MdM concluded, horizontal diffusion is anomalous, $x \sim t^{3/4}$. Note that the horizontal thermal noise η_h produces only a trivial additive contribution, which furthermore can be neglected at large time $(t \gg D D_h^2 \sigma^{-2})$. We thus set

$\eta_h = 0$ in the rest of the paper ($D_h = 0$). We also set $[V(z)] = 0$, since adding to $V(z)$ a horizontal bias V_0 amounts only to a trivial uniform translation

$$x(t) = V_0 t + x(t)|_{v_0 = 0}$$

If one thinks of a stratified porous medium where each layer has a slightly different permeability as a possible experimental system, the average velocity V_0 will be nonzero, then leading to anomalous *dispersion.*

Let us recall the physical mechanism responsible for anomalous diffusion in this model, which was pointed out in ref. 11. We will also use in this paper a discretized version of (1.1) (with $\eta_h = 0$) such that, by analogy with (1.2), the horizontal position of a given particle at integer time t reads

$$x(t) = \sum_{t'=1}^{t} V(z(t')) = \sum_{k=-\infty}^{+\infty} V(k)\, n(k, t) \tag{1.5}$$

where the vertical position $z(t)$ is now a symmetric random walk on a Z-lattice for which $n(k, t)$ is the total number of visits at site k between 0 and t. Since $x(t)$ is a sum of t random variables of zero mean, one could conclude that $x \sim t^{1/2}$. This is wrong, however, due to multiple visits to the same layer. Indeed, between 0 and t a one-dimensional walker visits only typically $\sim t^{1/2}$ distinct sites, each being visited $\sim t^{1/2}$ times. (5) can thus be rewritten as

$$x(t) \sim t^{1/2} \sum_{k \lesssim t^{1/2}} V(k)$$

The sum of $\sim t^{1/2}$ centered random variables $V(k)$ being of order $\sim t^{1/4}$, one has typically $x(t) \sim t^{3/4}$. The *spatial* long-range correlations of the medium thus generate *temporal* long-range correlations in successive elementary horizontal displacements of the walker, resulting in anomalous diffusion. This type of analysis, when applied to more general correlated environments,[7,12] gives useful "Flory" (or "dimensional") approximations for the diffusion exponent. It can be shown[12] that these approximations are in fact *exact* whenever the flow is divergenceless ($\partial_\mu V_\mu = 0$). This is also the case for the MdM model [since $V_z = 0$, $V_x = V(z)$], although here it takes a particuliarly transparent form due to the partial decoupling between the motion along x and z. This makes this model a particularly interesting way to approach more complicated situations.

Because of the apparent simplicity of this model one could hope to go beyond the simple calculation of the second moment (1.4) and obtain the behavior of a packet $P(x, z, t)$ of diffusing particles initially concentrated at $x = z = 0$ (Green function). This, however, is quite a formidable task. We are able, however, to compute the *average* front $[P(x, z, t)]$ for the continuum model (1.1), but only for $z=0$ (e.g., the return probability within a layer). This is equivalent to obtaining all moments $[\langle x^{2n}(t)\rangle_{z=0}]$. It is obtained by noticing that there is an exact relation between the Green functions of the MdM model (1.1) and of the localization problem in the 1D random potential $V(z)$, with Hamiltonian $H_\lambda = -D\nabla^2 + \lambda V(z)$. Formally, this relation reads

$$\int_{-\infty}^{+\infty} dx \int_0^{+\infty} dt\, e^{-st} e^{-\lambda x} P(x, z, t) = \langle 0| \frac{1}{s+H_\lambda} |z\rangle \tag{1.6}$$

(its correct meaning is given in Section 2). It is valid for *any* configuration of $V(z)$ and can be averaged over disorder. The case of a 1D Gaussian white noise potential [as in (1.4)] is one of the rare example where some exact results are available. Thus, using the well-known results[13-15] for the averaged Green function of the localization problem for coinciding points $z=0$ (no closed analytical expression exists for $z \neq 0$), we obtain for the MdM model (after a few transformations)

$$[P(x, z=0, t)] = at^{-5/4} Q(bxt^{-3/4}) \tag{1.7}$$

where a and b are constants. Two exact and complementary expressions are obtained for the scaling function $Q(y)$ using analytical properties of Airy functions. The result (1.7) is already very interesting. First, the important point is that its validity is not restricted to the Gaussian white noise potential case. Indeed, from (1.6) the limit $t, x \to \infty$ of this diffusion problem corresponds to the weak-disorder limit ($\lambda \to 0$) near the band edge ($s \to 0$) of the localization problem, which is known to be universal in the scaling region. Thus, $Q(y)$ is *universal.* Furthermore, we find that the constants a and b do not depend on microscopic details and can thus be computed exactly, using expressions like (1.4), for all the models within the universality class of the Gaussian distribution (e.g., potential with short-range correlations, etc.). This should allow for easy comparison with numerical simulations. We can also analyze the asymptotic behavior of $Q(y)$ and we find $[P(x, z=0, t)] \sim \exp[-C(x/t^{3/4})^{4/3}]$. Lifschitz arguments indicate this should remain true for $z \neq 0$.

One can also consider the MdM model in d_v "vertical" dimensions. Although diffusion becomes normal, $x \sim t^{1/2}$, in $d_v > 2$, it is easy to see that higher-order moments do have anomalous behavior in higher dimensions. The question of the diffusion front is intriguing. Flory–Lifschitz arguments indicate that it becomes Gaussian for $d_v > 2$ with, however, nontrivial tails. Finally, although the averaged diffusion front $[P(x, z, t)]$ is a physically interesting quantity, it is different from the *typical* packet in a given environment. This phenomenon was discussed in ref. 17.

1.2. Decay of the Magnetization of Diffusing Spins

The transverse magnetization $M = M_{x_1} + iM_{x_2}$ of a magnetic moment in a magnetic field H applied along a third orthogonal direction x_3 rotates with a frequency $\omega_0 = \gamma H$ (Larmor precession). If the field is slightly inhomogeneous, different magnetic moments will rotate with different frequencies $\omega_0 + \omega(z)$ and the *total* transverse magnetization will thus decay. If in addition the moments diffuse in d dimensions (which coordinate we denote by z), the macroscopic local transverse magnetization satisfies the following equation:

$$\frac{\partial M(z, t)}{\partial t} = D\nabla_z^2 M(z, t) + i\omega(z)\, M(z, t) \tag{1.8}$$

[the uniform factor $\exp(i\omega_0 t)$ has been eliminated by transforming to the rotating frame]. In a recent work with Mitra,[18] we have considered the case where $\omega(z)$ is a random function, motivated by the fact that static quenched random fields do exist in some NMR experimental systems, such as porous rocks. Using a mapping of (1.8) onto the well-studied problem of the self-avoiding walk (polymer), we have obtained quite general expressions for the asymptotic decay of the magnetization.

Here we concentrate mostly on $d = 1$, and using the exact results of Thouless[15] for the $d = 1$ polymer, we obtain the exact form of the decay of the magnetization. For the *total* magnetization it reads at long times

$$\int dz\, [M(z, t)] \sim \exp\{-1.74... \sigma^{4/3}(4D)^{-1/3} t\} \tag{1.9}$$

As we discuss here, this formula has a wide range of validity and should also apply to discrete models, with appropriate definitions of σ and D which are indicated in Section 4. Note that a qualitatively correct analysis of the $d = 1$ case was given by the authors of ref. 19, who did not seem to be aware of the mapping onto a polymer: the exact result (1.9) should also apply to their discrete model.

The spin depolarization problem is also directly related to the MdM model: the total dephasing accumulated by a single diffusing magnetic moment is $\varphi(t) = i \int_0^t \omega(z(\tau))\, d\tau$ and is thus, apart from the factor i, identical to the *horizontal displacement* $x(t)$ in the MdM model. The tracers in the MdM model which did not diffuse much in the x direction correspond to the diffusing moments which did not accumulate too much phase and thus dominate the decay of the magnetization: as we will see, both further correspond to stretched, self-avoiding Brownian trajectories in the z coordinate!

1.3. Diffusion in the Presence of Random Sources and Sinks

This model has received some renewed attention recently.[20,21] In this model independent particles in concentration $P(z, t)$ diffuse and can be created or annihilated locally with a rate $V(z)$ (negative or positive, respectively):

$$\frac{\partial P(z,t)}{\partial t} = D\nabla_z^2 P(z,t) - V(z)\,P(z,t) \tag{1.10}$$

Note that $P(z, t)$ is also the partition function of an *ideal* chain in a random potential $V(z)$. Recently, (1.10) was studied for Gaussian continuum randomness in $d=1$, and the *average* local concentration $[P(0, t)]$ was obtained. Here we correct some errors in the result of a recent letter.[20] Since $[P(0, t)]$ is obviously related to the *density of states* $\rho(E)$ of the Hamiltonian $H = -D\nabla^2 + V$ through

$$[P(0,t)] = \int_{-\infty}^{+\infty} dE\, e^{-Et} \rho(E)$$

and that $\rho(E)$ has been known for a long time,[13,15] it was not necessary to rederive it in Ref. 20. Here we obtain it directly and very simply. As we will see there is also a connection between the long-time behavior is this problem and the region of the diffusion front of the MdM model which corresponds to *stretched* walks along x.

This paper is organized as follows. In Section 2 we discuss in great detail the connections between all these models, which are then used extensively in the following sections. In Section 3 we concentrate on the model of Matheron and de Marsilly in its original form (one transverse dimension). We obtain several expressions for the average diffusion front, discuss its universality, and obtain the moments $[\langle x^{2n}\rangle]$. In Section 4 we obtain new results for the spin depolarization problem in $d=1$ using the connection with the 1D polymer, discuss universality, and review some results that we obtained recently in ref. 18 in higher dimensions. In Section 5 we compute the average local concentration in the problem of diffusion in the presence of sources and sinks, and discuss its universality and its relation to the density of states of the localization problem. In Section 6 we use Flory–Lifschitz arguments to relate various tails of the scaling functions of these models in $d=1$ and in higher dimensions. In particular we obtain the exact expression of the front for the MdM model on a cylinder. Section 7 is the conclusion, where some of the more recent literature is reviewed. Finally a diagrammatic calculation of the moments $[\langle x^{2n}\rangle]$ is performed in the Appendix.

2. MAPPING ONTO A RANDOM POTENTIAL OR A POLYMER PROBLEM

In this section we study in detail the exact relations between the MdM model, the localization in a random potential, the polymer problem, the spin depolarization problem, and the problem of diffusion in the presence of sources and sinks. These mapping are very general and valid in any dimension and will be applied to particular cases in the following sections. An an example of application we solve the case of the Cauchy distribution for $V(z)$.

2.1. Relation between the MdM Model, the Localization in a Random Potential, and the Diffusion with Sources and Sinks

Let us start with the continuum version (1) of the Matheron and de Marsilly model (with $\eta_h = 0$). For simplicity we discuss here the case of $d_v = 1$ „vertical" dimension (coordinate z), but extension to arbitrary d_v is straightforward. For a given configuration of the flow field $V(z)$ we define $P(x, z, t)$ to be the probability of the presence at (x, z) at time t for a particle initially at the origin, i.e., satisfying $P(x, z, t=0) = \delta(x)\,\delta(z)$. We will also use its Laplace transform $P(x, z, s) = \int_0^{+\infty} P(x, z, t) e^{-st}\, dt$. There are several methods to relate $P(x, z, s)$ to the Green function of the Schrödinger equation in the potential $V(z)$.

The first method makes use of the Feynman–Kac path integral[22,23] in order to compute the generating function of the moments $\langle x^n(t) \rangle_x$:

$$P(\lambda, z, t) = \int_{-\infty}^{+\infty} dx\, e^{-\lambda x} P(x, z, t) = \left\langle \exp\left(-\lambda \int_0^t du\, V(z(u)) \right) \right\rangle_z \tag{2.1}$$

where $\langle \cdots \rangle_z$ denotes the unnormalized thermal average over all Brownian paths $z(u)$, $0 \leqslant u \leqslant t$, with $z(0) = 0$ and $z(t) = z$. In fact our problem is precisely the famous problem solved by Kac of finding the probability distribution of the variable $x(t) = \int_0^t V(z(u))\, du$. In ref. 22, Kac solved it for V positive, or at least bounded from below. Here we are interested in cases where V can be of arbitrary sign and arbitrarily large (although with vanishing probability), but this will be handled below using some analytical continuations. One can rewrite the generating function (2.1) as a path integral over Brownian paths:

$$P(\lambda, z, t) = \int_{\substack{z(0)=0 \\ z(t)=z}} Dz(u) \exp\left(-\int_0^t du \left\{ \frac{1}{4D} \left(\frac{dz}{du} \right)^2 + \lambda V(z(u)) \right\} \right) \tag{2.2}$$

which, as is well known,[22,23] is also equal to the Green function

$$P(\lambda, z, t) = \langle z | \exp(-H_\lambda t) | 0 \rangle \equiv G_\lambda(z, t) \tag{2.3}$$

of the Hamiltonian $H_\lambda = -D\,\partial^2/\partial z^2 + \lambda V(z)$ associated with the action in (2.2). By definition, $G_\lambda(z, t)$ satisfies the equation $\partial G_\lambda(z, t)/\partial t = -H_\lambda G_\lambda(z, t)$ with initial condition $G_\lambda(z=0, t) = \delta(z)$, and is thus equal to the local concentration after time t of a packet initially at $z=0$ diffusing in a medium with sources and sinks of strength $\lambda V(z)$. It is related through (2.3) to the generating function of the horizontal displacement of the MdM model. Note that in cases of interest here, the spectrum of H may extend to the region of arbitrarily large negative energies. Thus, for t and z fixed, (2.3) holds only if both expressions are finite but for most of the applications here this is the case (in particular for the Gaussian white noise), because $P(x, z, t)$ decays faster than exponentially when $x \to -\infty$ (z and t fixed).

From (2.3) one can obtain two useful relations between the MdM model and the localization problem. First one can average over disorder [or equivalently, for a given configuration, sum over initial positions $z(t=0)$ with $z = z(t) - z(0)$]. Using a formal expansion in terms of the eigenstates Ψ_n and eigenvalues E_n of H_λ, one has

$$[P(\lambda, z, t)] = \left[\sum_n \Psi_n(0)\, \Psi_n^*(z) \exp(-E_n t)\right] = \int_{-\infty}^{+\infty} dE\, \rho_\lambda(z, E) \exp(-Et) \tag{2.4}$$

with

$$\rho_\lambda(z, E) = \sum_n \Psi_n(0)\, \Psi_n^*(z)\, \delta(E - E_n) = \pm\pi^{-1}\, \mathrm{Im}[G_\lambda(z, E \mp i\varepsilon)]$$

and where $G_\lambda(z, E)$ is the Green function of the localization problem (2.5). In particular, $\rho_\lambda(z=0, E) \equiv \rho(E)$ is the *density of states.*

The second relation is obtained by taking the Laplace transform of (2.3) without averaging:

$$\int_{-\infty}^{+\infty} dx \int_0^{+\infty} dt\, e^{-st} e^{-\lambda x} P(x, z, t) = \langle 0 | \frac{1}{s + H_\lambda} | z \rangle \equiv -G_\lambda(z, E = -s) \tag{2.5}$$

This relation is true strictly speaking only for $\mathrm{Re}(s) > \mathrm{Max}_n(-E_n)$, which is infinite if $V(z)$ is, for instance, a Gaussian white noise. In that case the Green function $G(z, E)$ defined by (2.5), which can be continued in the entire complex plan in the standard way, has poles on the entire real axis (for a given configuration) and (2.5) is only formal for s real. Once averaged over disorder, $G(z, E)$ has an imaginary part and a cut along the entire real axis [see the definition of $\rho_\lambda(z, E)$ above]. It is possible, however, to deduce a useful relation from (2.5), for instance, *by analytic continuation in* λ. After the average it reads

$$\int_{-\infty}^{+\infty} dx \int_0^{+\infty} dt\, e^{-st} e^{-i\lambda x}[P(x, z, t)] = -[G_{i\lambda}(z, -s)] \tag{2.6}$$

where both sides are real (in the absence of bias $[P(x, z, t)]$ is an even function of x) and well defined (for $s > 0$) in all the cases that we study. As we discuss in the next subsection, (2.6) is the partition function of a polymer.

Since we will use extensively formulas (2.4) and (2.6) in the following, let us check them in the case $\lambda = 0$. One then obtains as expected that the normalization $P(\lambda = 0, z, t) \equiv \int dx\, P(x, z, t)$ of the probability distribution inside a fixed layer z is equal to the free propagator along z:

$$P_0(z, t) = (4\pi Dt)^{-1/2} \exp[-z^2/(4Dt)]$$

Indeed, in (2.5) the Hamiltonian $H_0 = -D\nabla^2$ then corresponds to the free diffusion along z. For a free particle one has that

$$\rho_0(z, E) = (2\pi)^{-1}(ED)^{-1/2} \cos[|z|\,(E/D)^{1/2}], \qquad E > 0$$

and

$$\rho_0(z, E) = 0, \qquad E < 0$$

Then Eq. (2.4) correctly leads to $P(\lambda = 0, z, t) = P_0(z, t)$ and Eq. (2.5) leads to

$$\int dx\, P(x, z, s) = P_0(z, s) = (4sD)^{-1/2} \exp[-|z|\,(s/D)^{1/2}]$$
$$\equiv -G_0(z, -s) \qquad \text{for} \quad s > 0$$

One can also recover (2.5) without using the Feynman–Kac formula through an explicit expansion of the moments $\langle x^n(t) \rangle$ in powers of λV. This is done in the Appendix. Note that this expansion is equivalent, order by order, to the weak-disorder expansion in localization and gives a correct meaning to the analytic continuation in λ.

2.2. Relation with a Polymer Problem and with the Depolarization of Diffusing Spins

It is well known since the work of Edwards[24] and Thouless[15] that the *averaged* Green function of an electron in a random potential is related to the correlation function of a polymer by an analytic continuation. This continuation is precisely $\lambda \to i\lambda$, which transforms (2.5) into (2.6). It is also interesting to map directly the MdM model on a polymer problem. Let us consider first the discrete model (1.5) in the case where the $V(k)$, k integer, are centered Gaussian independent random variables of variance σ. The horizontal displacement is given by $x(t) = \sum_k V(k)\, n(k, t)$, where $n(k, t)$ is the number of visit at site k from between 0 and t. The *Fourier transform* of the average diffusion front can then be related to the partition function of a polymer. Averaging first over disorder for a *fixed* path $z(u)$,

$$\int dx \exp(-i\lambda x)[P(x, z, t)] = [\langle \exp(-i\lambda x(t)) \rangle_z]$$
$$= \left\langle \exp\left(-\lambda^2 \frac{\sigma}{2} \sum_k n(k, t)^2\right)\right\rangle_z \qquad (2.7)$$

where $\langle \cdots \rangle_z$ means that the endpoints are kept fixed, $z(0) = 0$, $z(t) = z$ (and satisfies $\int dz\, \langle 1 \rangle_z = 1$). The last term in (2.7) is precisely the partition function for a chain of t segments with repulsive self-interaction and fixed ends (e.g., its correlation function). The energy associated to n visits of the same site is positive and proportional to n^2 (this corresponds to the so-called Joyce model). For a continuum Gaussian potential, (2.7) has the same form (with $\sum_t$ replaced by $\int_0^\infty dt$), the partition function being replaced by

$$\left\langle \exp\left(-\lambda^2 \frac{\sigma}{2} \int_0^t \int_0^t du\, dv\, \delta(z(u) - z(v))\right)\right\rangle_z \qquad (2.8)$$

Note that the Laplace transform of (2.7) with respect to time, equal to $-[G_{i\lambda}(z, -s)]$ from (2.6), is equal to the same partition function in the grand canonical ensemble (s being the chemical potential). Finally, also note that the average concentration in the presence of sources and sinks $[G_\lambda(z, t)]$ has an expression analogous to (2.7) with $\lambda^2 \to -\lambda^2$ and can thus be interpreted as a polymer with *attractive* self-interaction.

The relation with the spin depolarization problem, which was also presented in ref. 18, is the following. The averaged transverse magnetization for n steps of a spin performing a discrete random walk in a Gaussian random magnetic field inducing random frequencies $\omega_k = \lambda V(k)$ (see Section 1) is also

$$M(z, t) = \left[\left\langle \exp\left(i\lambda \sum_k V(k, t) \right) \right\rangle_z \right]$$
$$= \left\langle \exp\left(-\frac{1}{2} \sigma \lambda^2 \sum_k n(k, t)^2 \right) \right\rangle_z \tag{2.9}$$

where the sum in the exponent of the r.h.s. is the total phase accumulated by the spin after n steps {note that $\lambda^2 \sum_k n(k, t)^2$ would be replaced by $\sum_k F[\lambda n(k, t)]$ for a more general distribution $P(\omega)$ of random frequencies such that $\int d\omega\, P(\omega) \exp(in\omega) = \exp[-F(n)]$}. The *total* magnetization $M(t) = \int dz\, M(z, t)$ is thus equal to the unrestricted partition function of a self-repulsive walk. It *decays* with time because it is normalized to the partition function of a free chain.

To conclude this subsection, let us give a simple application of the mapping onto the polymer. In the case where the $V(k)$ are independent random variables with a Cauchy distribution $P(V) = m/\pi(m^2 + V^2)$, it allows a very quick calculation of $[P(x, z, t)]$, because the self-interaction of the chain vanishes. One has then

$$[\langle \exp(-i\lambda x(t)) \rangle_z]$$
$$= \left\langle \exp\left(-|\lambda| m \sum_k |n(k, t)| \right) \right\rangle_z = \exp(-|\lambda| mt)\, P_0(z, t) \tag{2.10}$$

thus leading to a solution where horizontal and vertical motions are totally decoupled:

$$[P(x, z, t)] = \frac{mt}{\pi(m^2 t^2 + x^2)} P_0(z, t) \tag{2.11}$$

Now, using (2.6), we have

$$G_{i\lambda}(z, -s) = -\int ds\, e^{-st} e^{-\lambda mt} P_0(z, t) = -P_0(z, s + \lambda m) \tag{2.12}$$

performing back the analytical continuation $\lambda \to i\lambda$, one obtains exactly Llyod's result[25] for the average Green function in a random potential with a Cauchy distribution. The present method, however, is quite fast. It is also obvious that $P_0(z, t)$ can be replaced by *any* translationally invariant and Markovian diffusion process on the vertical axis. Since $\langle x(t) \rangle$ is typically $\sim mt$, the Cauchy distribution belongs to a different universality class than the Gaussian which is studied in Section 3.

2.3. Generalizations of the MdM Model

Here we indicate another and more direct method to derive equations like (2.4)–(2.6), which is very suitable to a generalization of the model. One starts from the Fokker–Planck equation associated to (1.1):

$$D\frac{\partial^2 P}{\partial z^2}+D_h\frac{\partial^2 P}{\partial x_2}-V(z)\frac{\partial P}{\partial x}=\frac{\partial P}{\partial t} \qquad \text{with} \quad P(x, z, t=0)=\delta(x)\,\delta(z) \tag{2.13}$$

Setting $D_h=0$, multiplying by $e^{-\lambda x}$, and integrating over x, one obtains the following equation for $P(\lambda, z, t)$:

$$D\frac{\partial^2 P}{\partial z^2}-\lambda V(z)P=\frac{\partial P}{\partial t} \qquad \text{with} \quad P(\lambda, z, t=0)=\delta(z) \tag{2.14}$$

which is formally equivalent to (2.5)–(2.6). The Fourier transform $P(i\lambda, z, t)$ satisfies a similar equation with $\lambda \to i\lambda$.

From (2.14) one can study the effect of boundary conditions. For instance, one can impose periodic boundary conditions along the x axis by closing it into a circle of length L. Each flow line closes on itself. The total probability on this cylinder $P_L(x, z, t)$ at $0<x<L$ is then simple equal to

$$P_L(x, z, t)=\sum_{n=-\infty}^{+\infty} P(x+nL, z, t)$$

where P satisfies (2.13). Thus P_L can be expressed as a discrete sum of Fourier modes involving the Fourier transform of P:

$$P_L(x, z, t)=L^{-1}\sum_{n=-\infty}^{+\infty} \exp\left(\frac{2i\pi n}{L}x\right) P\left(\lambda=\frac{2i\pi n}{L}, z, t\right) \tag{2.15}$$

We have been unable, however, to obtain a simple expression of the similar problem with *absorbing* boundary conditions $P(x, z, t)=0$ at $x=+L/2$ and $x=-L/2$. This is because the usual image method fails since (2.13) is not Hermitian.

Formula (2.14) allows for generalizations. Obviously, if the operator $D\,\partial^2 P/\partial z^2$ in (2.13) is replaced by *any diffusion operator* $\mathbf{O}_z P$ (discretized or continuous, random or homogeneous,...), one sees that (2.14) still holds with H_λ replaced by $H_\lambda=-\mathbf{O}_z+\lambda V(z)$. In particular, any choice of $\mathbf{O}_z$ such that the density of states or the random Hamiltonian is known provides exact results for $[P(x, z=0, t)]$. In a related work[26] we study the case where $\mathbf{O}_z$ is the diffusion operator in a vertical random force field (generalizing the Sinai problem). In this paper we will also consider the

MdM model with d_v "vertical" dimensions and d_h "horizontal" dimensions. Of particular physical interest are the cases $d_v = 1$, $d_h = 2$ and $d_v = 2$, $d_h = 1$. This model is described by (1.1) where x now a d_h-dimensional vector, z a d_v-dimensional vector, and the random velocity flow $V(z)$ is a horizontal d_h-dimensional vector, a function only of the d_v vertical coordinates. Instead of (2.1) one then studies

$$P(\lambda, n, z, t) = \langle \exp(-\lambda n \cdot x(t)) \rangle_z \tag{2.16}$$

where n is a horizontal d_h-dimensional unit vector. Taking $\mathbf{O}_z$ to be the free diffusion operator in d_v dimensions ∇_z^2, one obtains

$$P(\lambda, n, z, s) = \langle 0 | \frac{1}{s - \mathbf{O}_z + \lambda n \cdot V} | z \rangle \tag{2.17}$$

3. AVERAGED DIFFUSION FRONT FOR THE MODEL OF MATHERON AND DE MARSILLY

In this section we study the model of Matheron and de Marsilly of diffusion in a random layered flow in more detail. We first consider the continuum version (1.1) of this model in two dimensions with a velocity field $V(z)$ distributed according to a Gaussian white noise process. As discussed in Section 2, it can be mapped onto the problem of an electron in the random potential $\lambda V(z)$. Since the work of Thouless, the analytic properties of the averaged Green function of the corresponding Hamiltonian H_λ *at coinciding points* $[G_\lambda(z=0, E)]$ are known. This allows for an exact determination of $[P(x, z=0, t)]$. To our knowledge, the Green function, at noncoinciding points, does not have a simple expression, which seems to forbid an exact determination of $[P(x, z, t)]$ for $z \neq 0$. Some asymptotic results can, however, be obtained. We have used two complementary methods to obtain the function $[P(x, z=0, t)]$ by double Fourier–Laplace inversion of (2.6). The first method gives an integral representation of the scaling function in terms of the density of states of the localization problem, while the second method gives a series expansion. The asymptotic form of the diffusion front in the region $x \gg t^{3/4}$ is obtained and shown to be related to the Lifschitz tail of the random potential. Finally, we study the question of the universality of the average diffusion front and show that at large time a large class of discrete models have the same scaling function as the continuum model. It is an interesting particularity of this model that it is even possible to obtain very simply all coefficients for any discrete model.

3.1. Integral Equations Satisfied by $[P(x, z=0, t]$ and Scaling Function $Q(y)$

Let us first summarize the known results for the 1D Hamiltonian $H_\lambda = -D\,\partial^2/\partial z^2 + \lambda V(z)$, where $V(z)$ is a continuum Gaussian random potential with $[V(z)]=0$, $[V(z)\,V(z')]=\sigma\delta(z-z')$, and zero higher-order connected moments.

From refs. 14 and 15 the density of states $\rho(E)$ extends from $-\infty$ to $+\infty$ and reads

$$\rho(E)=\frac{\partial}{\partial E}\left\{\frac{2\sigma^{1/3}\lambda^{2/3}}{\pi^2(4D)^{2/3}\,M^2(E(4D)^{1/3}\sigma^{-2/3}\lambda^{-4/3})}\right\} \tag{3.1}$$

where $M^2(z)=Ai^2(-z)+Bi^2(-z)$ is the square of the *modulus* of the Airy function.[27] Note that using the large positive-z behavior $M^2(z)\sim\pi z^{-1/2}$, one recovers the free density of states $\rho(E)=(2\pi)^{-1}(ED)^{-1/2}\theta(E)$ from (3.1) in the limit $\lambda\to 0$. From ref. 15 the Green function equals

$$[G_\lambda(z=0,E)] = -\frac{\sigma\lambda^2}{2D}\frac{\partial^2}{\partial E^2}\operatorname{Ln} Ai(E(4D)^{1/3}\sigma^{-2/3}\lambda^{-4/3}e^{i\pi/3}) \tag{3.2}$$

Note that the imaginary part of (3.2) must be equal to $\pi\rho(E)$, which can be checked using (3.1) and

$$2e^{i\pi/3}\,Ai(ze^{i\pi/3})=Ai(-z)+i\,Bi(-z)\equiv M(z)e^{i\theta(z)} \quad\text{and}\quad \partial\theta/\partial z=\pi^{-1}M^{-2}$$

The result (3.2) was obtained by Thouless as an analytic continuation of the result (3.3) for the partition function $[G_{i\lambda}(z=0,E)]$ of the polymer problem (2.7), (2.8), itself obtained from the study of the $n=0$ component Φ^4 theory in $d=1$ (e.g., zero-dimensional quantum mechanics). From (2.6) this implies the following equation for the Laplace–Fourier transform of the diffusion front:

$$\begin{aligned}\int_{-\infty}^{+\infty} dx\, e^{-i\lambda x}[P(x,0,s)] &= -[G_{i\lambda}(z=0,-s)]\\ &= \frac{-\lambda^2\sigma}{2D}\frac{\partial^2}{\partial s^2}\operatorname{Ln} Ai(s(4D)^{1/3}(\sigma\lambda^2)^{-2/3})\end{aligned} \tag{3.3}$$

From the symmetry $x\to -x$ both sides of the equation are even functions of λ.

Before explicitly inverting the integral equation (3.3) to obtain $[P(x,z=0,t)]$, it is useful to make the scaling apparent by introducing scaling functions of dimensionless quantities. Notice that (3.3) can be written as

$$\int_{-\infty}^{+\infty} dx\, e^{-i\lambda x}[P(x, 0, s)] = \frac{a}{bs^{1/2}}\, G\left(\frac{\lambda}{bs^{3/4}}\right) \tag{3.4a}$$

$$G(v) = -2\pi^{1/2}|v|^{-2/3} H(|v|^{-4/3}) \qquad \text{with} \quad H(z) \equiv \frac{\partial^2}{\partial z^2} \operatorname{Ln} Ai(z) \tag{3.4b}$$

$$a = (\pi\sigma)^{-1/2}(4D)^{-1/4}, \qquad b = \frac{(4D)^{1/4}}{\sigma^{1/2}} \tag{3.4c}$$

It thus results that $[P(x, z=0, t)]$ is of the form

$$[P(x, z=0, t)] = \frac{a}{t^{5/4}}\, Q\left(\frac{bx}{t^{3/4}}\right) \tag{3.5a}$$

with

$$a = (\pi\sigma)^{-1/2}(4D)^{-1/4}, \qquad b = \frac{(4D)^{1/4}}{\sigma^{1/2}}, \qquad \int_{-\infty}^{+\infty} du\, Q(u) = 1 \tag{3.5b}$$

and $Q(-u) = Q(u)$. Note that integrating (3.5) over x gives $(a/b)t^{-1/2}$, which is the correct normalization since $a/b = (4\pi D)^{-1/2}$. The relation between $Q(u)$ and $G(v)$ is as follows: The Laplace transform $[P(x, z=0, s)]$ can be expressed as

$$[P(x, 0, s)] = a(b|x|)^{-1/3}\, F(s(b|x|)^{4/3}) \tag{3.6a}$$

with

$$F(u) = \int_0^{\infty} d\tau\, e^{-u\tau}\tau^{-5/4} Q(\tau^{-3/4}) \tag{3.6b}$$

{one checks that $\int dx\, [P(x, 0, s)] = (a/b)\pi^{1/2}s^{-1/2} 2 \int_0^{+\infty} du\, Q(u) = (4Ds)^{-1/2}$, as expected}. Fourier transforming, one obtains

$$G(v) = \int_{-\infty}^{+\infty} d\xi\, |\xi|^{-1/3} F(|\xi|^{4/3}) e^{-iv\xi} \tag{3.7}$$

From (3.6b), (3.7) one finds that $G(0) = \pi^{1/2}$ is equivalent to the normalization condition (3.5b). One then checks that it is consistent with (3.4b), since $\operatorname{Lim}_{z \to +\infty} z^{1/2} H(z) = -1/2$ [using the standard asymptotic expansions of the Airy function; see ref. 27, Eqs. (10.4.59)–(51)].

To exhibit the characteristic lengths more clearly, one can define $z = a_0 \hat{z}$, $t = \tau_D \hat{t}$, where a_0 is the cutoff in the z direction and $\tau_D = a_0^2/D$ is the natural unit of time. Then σ can be written as $\sigma = v^2 \xi$, where v is a local velocity and ξ is the correlation length of the random flow in the z direction. Using the natural units for x as $x = v\tau_D \hat{x}$, we find that formula (3.5) becomes in terms of the dimensionless quantities $\hat{x}$, $\hat{z}$, $\hat{t}$

$$\hat{P}(\hat{x}, \hat{z}=0, \hat{t}) = (2\pi)^{-1/2} \hat{t}^{-5/4} \left(\frac{2a_0}{\xi}\right)^{1/2} Q\left[\left(\frac{2a_0}{\xi}\right)^{1/2} \hat{x}\hat{t}^{-3/4}\right] \tag{3.8}$$

We will not work with these dimensionless quantities in what follows.

In the next subsection we invert (3.7) first and then (3.6b) to obtain an integral representation of $Q(u)$. In the Section 3.3 we do things in the opposite order by first Laplace inverting (3.3) and then Fourier transforming, obtaining a small-u series expansion for $Q(u)$.

3.2. Integral Representation of the Scaling Function Q

The function $G(v)$ is obtained from $Q(u)$ through two successive transformations, the first one, (3.6b), involving a Laplace transform and the second one, (3.7), a Fourier transform. The latter is easily inverted, and one obtains

$$\begin{aligned} F(u) &\equiv \int_0^{+\infty} d\tau\, e^{-u\tau} \tau^{-5/4} Q(\tau^{-3/4}) \\ &= -\frac{2}{\pi^{1/2}} u^{1/4} \int_0^{+\infty} dv \cos(vu^{3/4})\, v^{-2/3} H(v^{-4/3}) \end{aligned} \tag{3.9}$$

In order to perform the inverse Laplace transform of $F(u)$, a possible method is to transform the function $u^{1/4}\cos(vu^{3/4})$ into $u^{1/4}\exp(-cvu^{3/4})$ by rotating the domain of integration over v in the complex plane, and then to inverse Laplace transform in u. One has

$$F(u) = -\frac{2}{\pi^{1/2}} u^{1/4} \operatorname{Re}\left\{\int_0^{+\infty} dv\, e^{ivu^{3/4}} v^{-2/3} H(v^{-4/3})\right\} \tag{3.10}$$

$$= -\frac{2}{\pi^{1/2}} u^{1/4} \int_0^{+\infty} dv\, v^{-2/3} e^{-vu^{3/4}} \operatorname{Re}\{e^{i\pi/6} H(e^{-2i\pi/3} v^{-4/3})\} \tag{3.11}$$

(3.11) being deduced from (3.10) by the rotation $v \to iv$ of the integration half-axis of the complex integral in (3.10) and taking the real part at the end of the calculation. Using the definition (3.4b) of H and the properties of the Airy function, one also has

$$\mathrm{Re}\{e^{i\pi/6}H(e^{-2i\pi/3}z)\} = \mathrm{Re}\left\{-i\frac{\partial^2}{\partial z^2}\mathrm{Ln}(Ai(z)+i\,Bi(z))\right\}$$
$$= \frac{\partial}{\partial z}\left\{\frac{1}{\pi M^2(-z)}\right\} \tag{3.12}$$

Thus one has

$$F(u) = \frac{2}{\pi^{1/2}}u^{1/4}\int_0^{+\infty} dv\, e^{-vu^{3/4}}v^{-2/3}\frac{\partial}{\partial z}\left\{\frac{1}{\pi M^2(z)}\right\}\bigg|_{z=-v^{-4/3}} \tag{3.13}$$

Note that the function being integrated is exactly the (rescaled) density of states given by (3.1). We have also checked numerically that (3.9) and (3.13) are identical.

One can then introduce the inverse Laplace transform $g_v(\tau)$ of $u^{1/4}\exp(-vu^{3/4})$ obtained by contour integration along the imaginary axis or any path $z=re^{i\varphi}$ such that $\mathrm{Re}(z)\leqslant 0$ [if in addition $\mathrm{Re}(z^{3|4})>0$ the convergence is better]. We have

$$u^{1/4}e^{-vu^{3/4}} = \int_0^{+\infty} d\tau\, e^{-u\tau}g_v(\tau) \tag{3.14a}$$

$$g_v(\tau) = \frac{1}{\pi}\int_0^{+\infty} dr\, r^{1/4}\,\mathrm{Im}\{\exp[\tau re^{i\varphi} - vr^{3/4}e^{3i\varphi/4} + 5i\varphi/4]\} \tag{3.14b}$$

As we checked, the last expression is independent of the choice of $2\pi/3\leqslant\varphi\leqslant\pi/2$ and converges reasonably well for numerical purposes. The comparison of (3.9)–(3.13) and (3.14a), (3.14b) gives

$$\tau^{-5/4}Q(\tau^{-3/4}) = \frac{2}{\pi^{1/2}}\int_0^{+\infty} dv\, g_v(\tau)v^{-2/3}\frac{\partial}{\partial z}\left\{\frac{1}{\pi M^2(z)}\right\}\bigg|_{z=-v^{-4/3}} \tag{3.15}$$

which can be rewritten

$$\tau^{-5/4}Q(\tau^{-3/4}) = \frac{-3}{2\pi^{1/2}}\int_0^{\infty} dz\,\frac{\partial}{\partial z}\left\{\frac{1}{\pi(Ai(z)^2+Bi(z)^2)}\right\}g_1(\tau z)$$

and finally, since $g_1(0)=0$, we obtain the following integral representation for the scaling function Q:

$$\tau^{-4/5}Q(\tau^{-3/4}) = \frac{3}{2\pi^{1/2}}\int_0^{\infty} dz\,\frac{1}{\pi(Ai(z/\tau)^2+Bi(z/\tau)^2)}g_1'(z)$$

with

$$g_1'(z) = \frac{1}{\pi}\int_0^{+\infty} dr\, r^{5/4}\,\mathrm{Im}\{\exp[zre^{i\varphi} - r^{3/4}e^{3i\varphi/4} + 9i\varphi/4]\} \tag{3.16}$$

Asymptotic Shape of the Scaling Function Q for Large Arguments. From the fact that the Laplace transform of $g_1'(z)$ is $u^{5/4}\exp(-u^{3/4})$ one easily derives through saddle point methods the following asymptotic behavior for $g_1'(z)$ for small z:

$$g_1'(z) \underset{z\to 0}{\simeq} \frac{2187\cdot 2^{1/2}}{16384\pi^{1/2}} z^{-15/2}\exp\left(-\frac{3^3}{4^4}z^{-3}\right) \tag{3.17}$$

and for large z,

$$g_1'(z) \sim \frac{5}{16\Gamma(3/4)} z^{-9/4}$$

The behavior of $Q(y)$ for large y can be obtained from the saddle point at small z, large τ (large z/τ) in the integral (3.16) using

$$M^{-2}(-z) \underset{z\to\infty}{\simeq} \pi z^{1/2}\exp(-4z^{3/2}/3)$$

The result is

$$Q(y) \underset{y\to\infty}{\simeq} 3(2\pi)^{-1/2}|y|\exp[-(3y)^{4/3}/4] \tag{3.18}$$

The result shows that the large-deviation exponent is $\delta = 4/3$. We will discuss further the origin of this behavior in Sections 5 and 6.

3.3. Series Representation of the Scaling Function $Q(y)$

In this subsection we will first inverse Laplace transform Eq. (28):

$$\int_{-\infty}^{+\infty} dx\, e^{-i\lambda x}[P(x,0,s)] = \frac{-\lambda^2\sigma}{2D}\frac{\partial^2}{\partial s^2}\text{Ln}\, Ai(s(4D)^{1/3}(\sigma\lambda^2)^{-2/3}) \tag{3.19}$$

To do that, we first notice that the function $Ai'(z)/Ai(z)$ is analytic with all poles on the negative axis corresponding to the zeros of the Airy function. Those zeros are all simple and occur at $z = -a_n$, $n = 1, 2,..., \infty$, $a_0 = 2.338...$, where a_n is an increasing sequence. Since the inverse Laplace transform of $f(s)$ is

$$\frac{1}{2i\pi}\int_{\gamma - i\infty}^{\gamma + i\infty} f(s)e^{ts}\,ds$$

where γ has a real part larger than any singularity of $f(s)$, we can choose $\gamma < 0$ and close the contour at infinity in the half-plane $\text{Re}(s) < 0$ where the integral converges. Then one is left simply with

$$\int_{-\infty}^{+\infty} dx\,\exp(-i\lambda x)\,[P(x,0,t)] = \frac{\sigma\lambda^2}{2D}t\sum_{n=1}^{\infty}\exp[-a_n(4D)^{-1/3}(\sigma\lambda^2)^{2/3}t] \tag{3.20}$$

a result which could as well have been obtained by Laplace inverting from the start:

$$\int_{-\infty}^{+\infty} dx\, e^{i\lambda x}[P(x, 0, s)] \equiv \frac{\sigma\lambda^2}{2D} \sum_{n=1}^{\infty} \frac{\partial}{\partial s} \left\{ \frac{1}{s + (4D)^{-1/3} (\sigma\lambda^2)^{2/3} a_n} \right\} \tag{3.21}$$

A lot of information is available on the zeros of the Airy function a_n. They can be written [ref. 27, p. 450, Eq. (10.4.94)] $a_n = h[3\pi(4n-1)/8]$, where the asymptotic expansion of $h(x)$ is well known,

$$h(x) \sim x^{2/3} \left(1 + \sum_{k=1}^{\infty} c_k x^{-2k}\right)$$

and the c_k are tabulated. Since $a_n \sim (3\pi n/2)^{2/3}$ for large n it is clear that the above expressions (3.20) and (3.21) are convergent for any $t > 0$, s. In the limit λ goes to 0 the summation in (3.20) can be replaced by an integral which can be evaluated explicitly, which shows that expression (3.20) converges to the free diffusion result $(4\pi Dt)^{-1/2}$ as it should for $\lambda \to 0$.

Thus we have obtained the interesting result that the averaged diffusion front of the MdM model, at a given wavevector λ, has a discrete relaxation spectrum with inverse relaxation rates $\tau_n \sim |\lambda|^{4/3} a_n$. Note that these are exactly the eigenenergies of a 1D quantum mechanical model (without disorder) where the potential is $V = +\infty$ for $x < 0$ and $V(x) = \lambda^2 x$ for $x > 0$ (see Section 4).

We can now Fourier transform (3.20) and find after a change of variable in each term of the series $u \equiv q(4D)^{1/4} a_n^{3/4} t^{3/4} \sigma^{1/2}$:

$$[P(x, 0, t)] = \frac{t\sigma}{2D} \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} \frac{du}{2\pi} \left(\frac{4D}{\sigma}\right)^{3/4} a_n^{-9/4} t^{-9/4} u^2$$
$$\times \exp\{-|u|^{4/3} + iu[(4D)^{1/4} x (a_n t)^{-3/4} \sigma^{-1/2}]\}$$

Thus $[P(x, 0, t)]$ can be written under the scaling form (3.5):

$$[P(x, 0, t)] = \frac{a}{t^{5/4}} Q\left(\frac{bx}{t^{3/4}}\right) \tag{3.22}$$

and we find the following expression for the scaling function $Q(y)$:

$$Q(y) = \pi^{-1/2} \sum_{n=1}^{\infty} a_n^{-9/4} R\left(\frac{y}{a_n^{3/4}}\right) \tag{3.23}$$

where the function $R(z)$ is defined by

$$R(z) = \int_{-\infty}^{+\infty} du\, u^2 \exp(iuz - |u|^{4/3}) \tag{3.24}$$

Remarkably, $R(z)$ is related to the Levy stable probability distribution $Q_{4/3}(z)$ through

$$R(z) = -2\pi \frac{d^2}{dz^2} Q_{4/3}(z) \tag{3.25}$$

Let us recall[28] that the Levy distributions $Q_\alpha(z)$ $(0 < \alpha \leqslant 2)$ are stable under convolution and are the limit distribution of sums of centered independent variables $z = n^{-1/\alpha}(x_1 + \cdots + x_n)$, $n \to \infty$, where the distribution of x_i falls like $1/|x_i|^{1+\alpha}$ for large x_i (they do not possess a second moment and thus do not obey the usual central limit theorem). Note, however, that, once summed over the a_n the properties of the Levy distribution (long tails, etc.) disappear and the resulting $Q(u)$ seems to be quite a regular function. Although this connection with $Q_{4/3}$ is intriguing, since Levy diffusion fronts are known to appear in other models of diffusion in random media where long-tailed distributions of trapping time appear under renormalization, its physical interpretation here, if any, is not straightforward.

Formula (3.23) can be used to obtain a series representation for $Q(y)$. From ref. 28 we know that

$$Q_\alpha(z) = (\pi\alpha)^{-1} \sum_{k=0}^{\infty} (-1)^k z^{2k} \Gamma[(2k+1)/\alpha]/(2k)!$$

where the series is convergent for all z when $1 < \alpha \leqslant 2$. From there we obtain the following series for $Q(y)$:

$$Q(y) = \frac{3}{2(\pi)^{1/2}} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(3k/2 + 9/4)}{(2k)!} \left(\sum_{n=1}^{\infty} a_n^{-(3k/2+9/4)} \right) y^{2k} \tag{3.26}$$

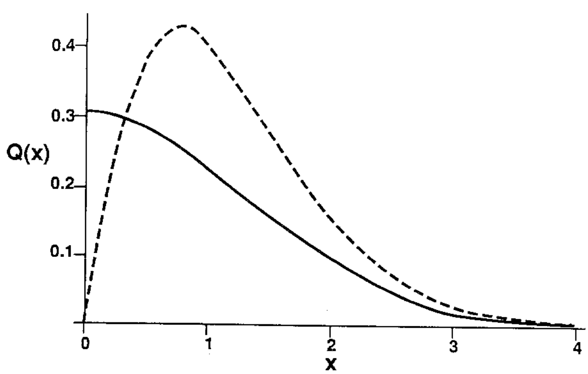


Fig. 2. Plot of the scaling function $Q(x)$ (solid line), obtained from the series expansion (3.26). The dashed curve is the stretched exponential (3.18), which becomes indistinguishable from $Q(x)$ for $x > 4$. The value $Q(0) = 0.306$.

This series is rapidly convergent everywhere and thus allows for a good determination of $Q(y)$. The small-y behavior is

$$Q(y) = 0.306007.. - 0.0425352.. y^2 + 0.001923546.. y^4 + \cdots$$

The function $Q(y)$ obtained by this method is plotted in Fig. 2, where one can see that it crosses over around $y \sim 4$ to the asymptotic expression (3.18). We have also checked the overall normalization.

3.4. Universality of the Result for the Diffusion Front

Thus far we have studied a continuum model with Gaussian uncorrelated disorder and we must ask whether it has anything to do with an actual simulation or experiment where (i) space and time could be discretized, (ii) the local velocity flow could have a non-Gaussian distribution, and (iii) there could be short-range correlations between velocities at different altitudes. It turns out that all these features are irrelevant for the *bulk* of the distribution $[P(x, 0, t)]$. More precisely, the statement is that, if $u = x/t^{3/4}$ is the rescaled variable,

$$\lim_{t \to \infty} t^{1/2}[\text{Prob}(y < u < y + dy)] = aQ(by)\, dy$$

Furthermore, since a and b can obviously be extracted from the normalization and from the second cumulant $[\langle x^2(t) \rangle]$, which can also be independently calculated for any model, a and b in fact do not depend on any microscopic details and are given by

$$a = (\pi\sigma)^{-1/2}(4D)^{-1/4}, \qquad b = (4D)^{1/4}\sigma^{-1/2}$$

where D is the asymptotic diffusion coefficient in the transverse direction and σ is defined by

$$\sigma = \int dz\, [V(0)\, V(z)]$$

and is thus equal to the zero-momentum Fourier component of the velocity autocorrelation, provided the integral exists (we have assumed $[V(z)] = 0$) and is thus independent of eventual higher-order cumulants of V. It is easy to check through the diagrammatic approach (see Appendix) that the fourth connected cumulant is irrelevant, for instance, provided

$$\int dz_1\, dz_2\, dz_3\, [V(0)\, V(z_1)\, V(z_2)\, V(z_3)]_{\text{conn}}$$

is finite, etc. The fact that the result depends only on these two simple infrared-only dependent parameters has to do with the so-called super-renormalizability of the corresponding field theory in $d=1$. Note that this property should extend up to, and including, $d_v=2$ transverse ("vertical") dimensions. As we will see, anomalous behavior can arise in the tails, but since $Q(y)$ is normalized to unity, the bulk contains almost all the tracer diffusing.

It is interesting to relate this universality of the diffusion front to the work of Derrida and Gardner,(16) who computed the DOS and localization length for a discrete 1D random potential λV_n through a weak-disorder expansion. They found universality for weak disorder in the vicinity of the band edge $E \to 0$, $\lambda \to 0$ with $E\lambda^{-4/3}$ fixed, which from the mapping of Section 2 corresponds exactly to the universality of diffusion at large time. This can also be seen from the $n=0$ quantum mechanics(15) using rescaling.

3.5. Expression for the Moments of the Displacement $[\langle x_{2k}(t)\rangle_0]$

From (3.3) one can extract an analytic expression for the moments of the displacement $[\langle x^{2k}(t)\rangle_0]$ for the Gaussian continuum model, where here $\langle\cdots\rangle_0$ denotes a *normalized* thermal average restricted to the walks which come back to the origin at t. In the Appendix a direct calculation of these moments is carried up to $2k=4$, and is found to agree with our general expression. As also discussed in the Appendix, for general discrete models the result below for the moments give the leading behavior at long time, where the coefficients D and σ defined.

Expanding (3.3) around $\lambda=0$ on both sides and using the asymptotic expansion of $Ai(z)$ for large z, as given in ref. 27 [p. 448, (10.4.59)], one easily obtains the Laplace transforms:

$$\int_{-\infty}^{+\infty} x^{2k}[P(x,0,s)] = (2k)!\,\frac{a}{bs^{1/2}}\,(-1)^k \beta_{2k}\,\frac{1}{b^{2k}s^{3k/2}}$$

with $\beta_0=\pi^{1/2}$, $\beta_2=-\pi^{1/2}/2$, and for $k \geqslant 1$,

$$\beta_{2k+2} = -2\pi^{1/2}\,\frac{3k}{2}\left(\frac{3k}{2}+1\right)\alpha_k$$

where the α_k are defined as the coefficients of the following series expansion:

$$\mathrm{Ln}\left(1+\sum_{k=1}^{\infty}(-)^k c_k\left(\frac{2}{3}\right)^{-k} x^k\right) \equiv \sum_{k=1}^{\infty}\alpha_k x^k$$

where the c_k enter the asymptotic expansion of the Airy function and are given by[27]

$$c_k = \frac{\Gamma(3k+1/2)}{54^k k!\,\Gamma(k+1/2)}$$

$c_0 = 1$, $c_1 = 15/216$, ect. Thus, inverse Laplace transforming and taking into account the normalization, we find for the moments

$$[\langle x^{2k}(t)\rangle_0] = (2k)!\,(-1)^k \beta_{2k} \sigma^k (4D)^{-k/2} \frac{1}{\Gamma((3k+1)/2)} t^{3k/2}$$

In particular we obtain

$$[\langle x^2\rangle_0] = \sigma\pi^{1/2}(4D)^{-1/2} t^{3/2}$$
$$[\langle x^4\rangle_0] = 4!\,(5\sigma/16)^2 (4\pi)^{1/2} (D\Gamma(7/2))^{-1} t^3$$

4. DECAY OF THE MAGNETIZATION OF DIFFUSING SPINS

Following the analysis of Section 2, the averaged local transverse magnetization $[M(z, t)]$ at site z of spins initially at $z=0$ diffusing in a Gaussian continuum random magnetic field is equal to the partition function $[G_{i\lambda}(z, t)]$ and of the 1D Edwards self-avoiding polymer (2.8) with ends fixed at 0 and z:

$$\begin{aligned}[M(z, t)] &= \int_{\substack{z(0)=0\\ z(t)=z}} Dz(\tau) \exp\left[-\int_0^t d\tau \left\{ \frac{1}{4D}\left(\frac{dz}{d\tau}\right)^2 \right.\right.\\ &\qquad \left.\left. + \frac{1}{2}\sigma\lambda^2 \int_0^t du\, \delta(z(u)-z(\tau)) \right\}\right]\\ &\equiv [G_{i\lambda}(z, t)]\end{aligned}$$

This relation is valid in any dimension [$z(\tau)$ becoming a d-dimensional vector].

In $d=1$ this quantity is known exactly at $z=0$ from (3.20), which was obtained through Laplace inversion of (3.19). Thus, the average magnetization of walkers returning to the same site $z=0$ is

$$[M(z=0, t)] = \frac{\sigma\lambda^2}{2D} t \sum_{n=1}^{\infty} \exp\{-a_n (4D)^{-1/3} (\sigma\lambda^2)^{2/3} t\} \qquad (4.1)$$

where the a_n are the absolute values of the zeros of the Airy function in increasing order. For short times this crosses over to the pure diffusion decay $(4\pi t)^{-1/2}$ as discussed in Section 3.3.

One has for long time

$$[M(z=0, t)] \sim \frac{\sigma\lambda^2}{2D} t \exp\{-2.34..(4D)^{-1/3}(\sigma\lambda^2)^{2/3} t\} \tag{4.2}$$

which corresponds to a transition at $s = s_c(\lambda) = 2.34..(4D)^{-1/3}(\sigma\lambda^2)^{2/3}$ in the grand canonical partition function of the polymer with endpoints constrained at $z = 0$, as was discussed by Thouless.[15]

The most interesting quantity, for experiments on one-dimensional systems is the averaged total magnetization $[M(t)] = \int dz\, [M(z, t)]$. This is because one can hope to measure the *disorder-averaged* magnetization only by averaging over many independent microscopic spins. In the thermodynamic limit the total magnetization becomes equal to the configurational average $[M(t)]$. To observe (4.1)–(4.2) one would have to select only the spins which came back to $z = 0$, which seems practically more difficult (except maybe by considering a large number of parallel chains and spins initially localized on a plane transverse to the chains).

The total magnetization corresponds to the unconstrained partition function of the 1D polymer (e.g., to the susceptibility of the $n = 0$ Φ^4 model). As was shown by Thouless,[15] the Laplace transform of this quantity diverges like $(s - s_c')^{-1}$ at a *different* critical "chemical potential" $s = s_c'(\lambda) = 1.74..(4D)^{-1/3}(\sigma\lambda^2)^{2/3}$. Thus we obtain at large time

$$[M(t)] \sim \text{const} \cdot \exp\{-1.74..(4D)^{-1/3}(\sigma\lambda^2)^{2/3} t\} \tag{4.3}$$

which has a slower decay rate than (4.2) (since the moments see the same field less often, e.g., the walks are more stretched).

4.1. Higher Dimension

In a related work[18] with Mitra we use the general relation between the spin depolarization problem and the self-avoiding walk problem (polymer) to obtain new results for the magnetization decay $[M(t)]$ in higher dimensions for the continuum Gaussian random magnetic field. In particular we obtain at large time

$$[M(t)] \sim t^{\gamma-1} e^{-s_c t}$$

where γ is the SAW susceptibility exponent and s_c the polymer free energy. For small λ we find that in $d \leqslant 2$, $s_c \sim \lambda^{4/(4-d)}$, $s_c \sim \lambda^2 \operatorname{Ln} \lambda$ in $d = 2$ and $s_c \sim \lambda^2$ in $d > 2$. Furthermore, the quantity $[M(z, t)]$ takes quite generally the scaling form for large t, $z \sim t^{\nu}$:

$$[M(z, t)] = t^{-\nu d} F(z t^{-\nu}) [M(t)]$$

where ν is the SAW correlation length exponent. Also, $[M(z=0, t)] \sim t^{\alpha-2} e^{-s_c t}$ (lattice cutoffs then have to be introduced). Note that $d=1$ is very particular.

4.2. Universality of the Results

The question of the universality of the continuum Gaussian model for the spin depolarization problem is analyzed in a very general framework in ref. 18. Here we make some remarks pertinent for the results (4.1)–(4.3) in $d=1$.

The property which is valid for a large class of discrete models (see below) is that the decay is exponential, and that the decay rate defined by

$$s_c(\lambda) = -\lim_{t \to +\infty} \frac{1}{t} \operatorname{Log} M(t)$$

[where $M(t)$ is either the quantity in (4.2) or (4.3)] is given for small λ exactly by the exponents in the expressions in (4.2)–(4.3), where, as in the MdM model, D and $\sigma\lambda^2$ are *independent of microscopic details.* Of course there will be a higher-order correction $\sim a\lambda^2$ to $s_c(\lambda)$ proportional to the lattice cutoff. Thus the exponential decays in (4.2)–(4.3) are *exact for any discrete model* in this universality class, where D is the actual diffusion coefficient measured at large t and $\sigma\lambda^2$ is the integrated correlation function of the second cumulant of the disorder, e.g., $\sigma\lambda^2 \equiv \int dz\, [\omega(0)\, \omega(z)]$. Thus, our result (4.3) should also apply to the discrete Gaussian model considered in ref. 19 through qualitative arguments. In addition to this strong universality, there is also a weaker property, namely that (4.1) gives the exact scaling function in the limit $t \to \infty$, $\lambda \to 0$, with $t\lambda^{4/3}$ fixed.

How big is this universality class? Let us consider a more general model in $d=1$ with a cutoff and a more general on-site distribution of magnetic field $P(\omega)$. In the $n=0$ quantum mechanics of Thouless this is equivalent to considering a more general potential $V(\Phi^2)$ defined through $\int d\omega \exp(i\omega n) = \exp(-V(n))$. For Gaussian randomness $V(\Phi^2) = \sigma(\lambda^2/2))(\Psi^2)^2$. Following the steps in ref. 15, it is easy to see that the calculation of $[M(z=0, s)]$ amounts to solving the following radial Schrödinger equation:

$$\left(-\frac{1}{D} \frac{\partial^2}{\partial \rho^2} + \frac{V(\rho)}{\rho} \right) \Phi_0(\rho) = (-s)\, \Psi_0(\rho) \tag{4.4}$$

where $\Phi_0(\rho)$ must decay to 0 at $\rho \to \infty$ and $\Phi_0(\rho=0)=0$. If E is the first eigenenergy of the Hamiltonian on the l.h.s., $s_c = -E$ is the first pole of $[M(z, s)]$, which then decays like $\exp(-s_c t)$. Clearly, if $V(n)$ is monotonic and grows at infinity faster than n, a simple rescaling $\rho \to \lambda^{-2/3}$, $s \to \lambda^{4/3}$ shows that only the behavior of $V(\rho) \sim \rho$ close to $\rho = 0$ is relevant for small disorder, and that the model belongs to the universality class of the Gaussian continuum model. However, if $V(\rho)/\rho$ goes to 0 at infinity, there will be only a continuum of eigenvalues accumulating at $s = 0$. The consequence is that in that case the magnetization will decay as a stretched exponential. This is discussed further in ref. 18.

One can also make a simple Flory argument which parallels the $n = 0$ quantum mechanics. The total weight associated with walks visiting s sites is

$$\exp(-Ds^2/t + sV(t/s)) \tag{4.5}$$

(s must not be confused with the Laplace variable!). Clearly $\rho = t/s$ and the exponent in (4.5) is exactly analogous to t multiplied by the operator on the l.h.s. if (4.4). The idea is that in $d = 1$, these Flory arguments are almost exact and that the $n = 0$ quantum mechanics gives the prefactors. Balancing terms in (4.5), one sees that $s \sim \lambda^{2/3} t$, and thus once again one checks that higher-order terms in $V(\rho)$ are irrelevant under the conditions discussed above. Note that the extension $R = s$ of an Edwards polymer with small self-repulsion is $R \sim \lambda^{2/3} t$.

5. SCALING FUNCTION FOR DIFFUSION IN MEDIA WITH SOURCES AND SINKS IN $D = 1$

As explained in Section 2.1, the *averaged* concentration at the origin in the model of diffusion in the presence of sinks and sources characterized by a strength λV, $[G_\lambda(z=0, t)]$, is directly related to the density of states $\rho_\lambda(E)$ of the Hamiltonian $H_\lambda = -D\nabla_z^2 + \lambda V(z)$ through

$$G_\lambda(z=0, t) = \int_{-\infty}^{+\infty} dE\, \rho_\lambda(E) \exp(-Et) \tag{5.1}$$

Here we compute it exactly for the Gaussian continuum model in $d = 1$ for the known expression of the density of states. We emphasize the relations with corresponding quantities in the MdM model.

From the expression (3.1) of the density of states, one obtains

$$G_\lambda(z=0, t) = A\lambda^{2/3} g(Bt\lambda^{4/3}) \tag{5.2}$$

with $A = (2/\pi^2)\sigma^{1/3}(4D)^{-2/3}$, $B = (4D)^{-1/3}\sigma^{2/3}$, and $g(u)$ defined by $(u \geq 0)$

$$g(u) = \int_{-\infty}^{+\infty} dz \exp(-zu) \frac{\partial}{\partial z} \left\{\frac{1}{M^2(z)}\right\} = u \int_{-\infty}^{+\infty} \exp(-z_u) \frac{1}{M^2(z)} \tag{5.3}$$

where $M(z)$ is the modulus of the Airy function.

In order to study $g(u)$ in various asymptotic regimes, we rewrite it as $g(u) = g_0(u) + g_+(u) + g_-(u)$, with

$$\begin{aligned} g_0(u) &= u \int_0^{+\infty} dz \exp(-zu)\, \pi z^{1/2} = \tfrac{1}{2}\pi^{3/2} u^{-1/2} \\ g_-(u) &= \int_0^{+\infty} dz \exp(-zu)\, \{M^{-2}(z) - \pi z^{1/2}\} \\ g_+(u) &= u \int_0^{+\infty} dz \exp(zu)\, M^{-2}(-z) \end{aligned} \tag{5.4}$$

where $g_0(u)$ corresponds to the free spectrum [obtained in (5.2) for $\lambda \to 0$ at fixed t] and thus to the decay due only to pure diffusion. $g_-(u)$ and $g_+(u)$ correspond to modifications of this behavior decreasing in time (negative frequences) and increasing (positive frequencies), respectively. In $g_-(u)$ the integrand goes to zero for $z \to +\infty$, since $M^{-2}(z) \sim \pi z^{1/2}$ in this limit. In $g_+(u)$, hgowever, the integrand has a maximum which gives the main contribution to $g(u)$ for $u \to +\infty$.

5.1. Small-Time Behavior

From Section 2.1, $G_\lambda(z=0, t)$ is the generating function of the moments $\langle x^n(t) \rangle$ of the horizontal displacement in the MdM model, studied in Section 3.5 and in the Appendix. Thus the small-u behavior of $g(u)$ is related to these moments. One has

$$g_+(u) + g_-(u) = u \int_{-\infty}^{+\infty} dz\, (M^{-2}(z) - \pi z^{1/2} \theta(z)) + O(u^2) \tag{5.5}$$

The integrand is a total derivative and using $M^{-2} = -\pi\, \partial\theta/\partial z$ with $\theta(+\infty) = \pi/2$, $\theta(z \to -\infty) = \pi/4 - 2z^{3/2}/3 + O(z^{-3/2})$ from ref. 27, p. 449, (10.4), one easily find

$$g(u) = \pi^{3/2} u^{-1/2}/2 + \pi^2 u/4 + O(u^2) \tag{5.6}$$

Once inserted back in (5.2) the first term gives the normalization, and the second gives back exactly the result of Section 3.5 for the second moment.

5.2. Long-Time Behavior

Using the asymptotic behavior for $z \to +\infty$, $M^{-2}(-z) \sim \pi z^{1/2} \exp(-4z^{3/2}/3)$ (e.g., the Lifschitz tail of the localization problem), we can obtain the asymptotic behavior of $g_2(u)$ for large u through a saddle point method and we obtain (the maximum is for $z^{3/2} = u^3/8$):

$$g(u) \sim g_2(u) \sim \tfrac{1}{2}\pi^{3/2} u^{5/2} \exp(u^3/12) \tag{5.7}$$

Restoring the factors from (5.2), we finally find the large-t behavior:

$$[G_\lambda(z=0, t)] \sim \frac{\lambda^4 t^{5/2}}{8(\pi D^3)^{1/2}} \exp\left(\frac{t^3 \lambda^4}{48D}\right) \tag{5.8}$$

Note that this result, which we believe is correct, differs by a factor of 6 from the result of Ref. 20 where these well-known results for the density of states in $d=1$ are unknowingly rederived.

5.3. Relation with the MdM Asymptotic Diffusion Front

Because of the relation

$$[G_\lambda(z=0, t)] = \int_{-\infty}^{+\infty} dE\, \rho_\lambda(E) \exp(-Et) = \int_{-\infty}^{+\infty} dx\, e^{\lambda x}[P(x, z=0, t)] \tag{5.9}$$

the asymptotic behavior (5.8) is related to the asymptotic shape of the front:

$$[P(x, z=0, t)] \sim \frac{3|x|}{\pi\sigma 2^{1/2} t^2} \exp\left(-\frac{3^{4/3} D^{1/3} x^{4/3}}{2^{4/3}\sigma^{2/3} t}\right) \tag{5.10}$$

and we have checked through a simple saddle point method that (5.8)–(5.10) are indeed compatible. Thus the stretched region $x \gg t^{3/4}$ of the front in the MdM model is related to the Lifschitz tails of the random potential.

5.4. Universality of the Result (5.2)

The universality of (5.2) is very limited. Compared to the other models in Sections 3 and 4, diffusion in the presence of random sources and sinks offers the least universality. For a model with cutoffs, (5.2) is true only at intermediate times, *only* in the limit $\lambda \to 0$, $t \to \infty$ with $t\lambda^{4/3}$ fixed. Since G_λ is the generating function of the moments $\langle x^n(t) \rangle$ and we know from Section 3 that these moments attain their value from the Gaussian model only for large time, it is clear that (5.2) is not correct at small time either for a discrete model (although the true behavior might then be easy to obtain perturbatively). On the other hand, at truly large time $t \gg \lambda^{-4/3}$, formula (5.2) and more precisely (5.8) crosses over to

$$[G_\lambda(0, t)] \sim \exp(C\lambda^2\sigma t^2) \tag{5.11}$$

coming from the single-site distribution. Using the analogy with the self-attracting walk discussed in Section 2.2, substituting $\lambda^2 \to -\lambda^2$ in the corresponding formula (2.7), one sees that the partition function is dominated by the configuration where the walk is collapsed at site 0, thus with $\sum_k n(k, t) = t^2$, leading to (5.11). This behavior was obtained by Zeldovich *et al.*[29] and the crossover between (5.8) and (5.11) was analyzed qualitatively very recently in ref. 21. Here we point out that the crossover from (5.8) to (5.11) corresponds in localization to the crossover from the Lifschitz tail $\rho(E) \sim \exp(-\text{const} \cdot E^{2-d/2})$ of the DOS for the continuum Gaussian model to the Lifschitz tail $\rho(E) \sim \exp(-\text{const} \cdot E^2)$ when the correlation length of the random potential exceeds the de Broglie thermal wavelength. For localization, this crossover was analyzed in ref. 30.

Finally, the behavior (5.11) corresponds to some tails of the diffusion front of the MdM model. It corresponds to ultrarare events of a particle staying in the same layer and to configurations where this layer has a very large velocity, and thus to a tail $\exp(-x^2/t^2)$ of the diffusion front. Note that (5.11) is very dependent on the shape of the distribution and thus highly nonuniversal.

6. TAILS, FLORY–LIFSCHITZ ARGUMENTS, AND FURTHER CONNECTIONS

In this section we analyze further the various tails in the diffusion front of the MdM model and their connections to the other models. There are two types: one can take the Laplace transform of the front [e.g., multiply by $\exp(-\lambda x)$ and integrate]: this emphasizes the horizontally stretched configurations $x \gg t^{3/4}$ and is connected to the Lifschitz tails of the localization model. Or, one can take the Fourier transform of the front, in which large horizontal displacements cancel and thus which emphasizes the tracers with small horizontal displacement. This is connected to spin depolarization and polymers.

6.1. Diffusion Front and Lifschitz Tails

We have obtained in Section 3 that the diffusion front $[P(x, z=0, t)]$ has a stretched exponential behavior $\exp(-\text{const} \cdot (x/t^{3/4})^{4/3})$ for $x \gg t^{3/4}$. We now give a qualitative argument for this behavior {which applies to

$\int dz\,[P(x, z, t)]$ as well}. This argument is of the Flory–Lifschitz type combining a saddle point analysis à la Lifschitz with dimensional argument à la Flory. We believe that it gives a reasonable indication of the behavior in arbitrary dimensions, although it neglects fluctuations.

Let us consider a discrete MdM model with d "vertical" dimensions. As discussed in Section 1, the horizontal displacement in a given flow configuration for walks which visit s *distinct* sites scales as

$$x \sim \frac{t}{s} \sum_{i=1}^{s} V(i) \tag{6.1}$$

and, being the sum of random variables, takes a Gaussian shape of variance t^2/s at large time. Note that the *typical* s is $s_{\text{typ}} \sim \inf(t^{d/2}, t)$ and thus $x_{\text{typ}} \sim t^{1-d/4}$ for $d<2$, $x_{\text{typ}} \sim t(\text{Ln}\, t)^{1/2}$ in $d=2$ and $x_{\text{typ}} \sim t^{1/2}$ in $d>2$.

On the other hand, the fraction of contracted walks which have $s \ll s^{\text{typ}} \sim \inf(t, t^{d/2})$ scales like $\exp(-Dt/s^{2/d})$. Thus, one expects that the diffusion front (both thermally and configurationally averaged) is like

$$[P(x, t)] \sim \int ds \exp(-Dt/s^{2/d} + sx^2/\sigma t^2) \tag{6.2}$$

The saddle point is at $s^* = (\sigma D t^3/x^2)^{d/(d+2)}$ and one finds

$$[P(x, t)] \sim \exp\left\{-\text{const} \cdot D^{d/d+2} \sigma^{-2/(d+2)} \left(\frac{x}{t^{1-d/4}}\right)^{4/(d+2)}\right\} \tag{6.3}$$

Note, however, that this is true only if $s^* \ll s_{\text{typ}} \sim \min(t, t^{d/2})$: thus in $d<2$ it is true only in the stretched region for $x \gg \sigma^{1/2} D^{-d/4} t^{1-d/4}$. Note the perfect agreement with (5.10) in $d=1$. For smaller values of $x/t^{(1-d/4)}$ the front is simply Gaussian, obtained by inserting $s = s_{\text{typ}} \sim (Dt)^{d/2}$ in (6.2) (stretched vertical walks $s \gg s_{\text{typ}}$ do not contribute appreciably here—see below, however). Now, for $d>2$ we find that the form (6.3) for the front *is still valid* in the ultrastretched regime

$$x \gg (\sigma D)^{1/2} (a^{d-2} D)^{-(d+2)/(2d)} t^{(1-1/d)}$$

which corresponds to the regime of validity that we expect for (6.2), e.g., $s \ll ta^{d-2} D$ (a cutoff a has to be introduced). This horizontally ultrastretched regime corresponds to ultracompressed vertical walks. For typical walks one obtains a Gaussian diffusion front for $d>2$:

$$\exp\left(-\text{const} \cdot \frac{Da^{d-2} x^2}{\sigma t}\right)$$

Note that in $d>2$, since we have still supposed $\eta_h=0$, the diffusion is normal only by virtue of the lattice cutoff a. In particular, $D_x \sim a^{2-d}$ diverges when $a \to 0$, as the simple one-loop integral would show (see Appendix).

A simpler but related argument is the following. Let us consider a binary distribution for V (it can be generalized). The probability of finding a region of size R^d in the vertical space such that $V=+1$ is $\exp(-R^d)$, and the probability that a walk remains there for time t is $\exp(-t/R^2)$. Thus, $[P(x \sim t, t)] \sim \exp(-t^{d/(d+2)})$ (survival trapping probability). If the front is $[P(x, t)] \sim \exp(-(x/t^{(1-d/4)})^{\delta})$, this implies $\delta = 4/(d+2)$ for $d<2$. This is correct. However, one could erroneously conclude from this argument in $d>2$ that $\delta = 2d/(d+2)$ if one was supposing a diffusion front of the type $\exp(-(x/t^{1/2})^{\delta})$ as would seem natural. This is wrong, as shown above, because there are additional length scales and the region $x \sim t$ becomes *atypical* above $d=2$.

One can make contact also with the conventional Lifschitz tail of the density of states of the Gaussian random potential $\rho(E) \sim \exp(-(\sigma\lambda^2)^{-1} E^{2-d/2})$ and check that one recovers exactly the diffusion front (6.3) using the correspondence (5.9) between the two models for $d<2$. This simple relation, however, breaks down in $d>2$.

6.2. A Further Relation between the MdM Model and the SAW: Periodic Boundaries

Let us impose periodic boundary conditions in the x direction by reinjecting a particle arriving at $x=L$ at $x=0$. Equivalently one can consider the MdM model on a cylinder, periodic in the x direction. Note that this does not affect the flow, each layer being closed on itself. Using Eq. (2.15) derived in Section 2, one can relate the averaged probability $[P_L(x, z, t)]$ for a fixed L to a sum of partition functions for a self-repelling chain of strength $\lambda_k = 2\pi k/L$:

$$[P_L(x, z, t)] = \frac{1}{L} \sum_{k=-\infty}^{+\infty} 2\cos\left(\frac{2\pi k x}{L}\right) [G_{i\lambda_k}(z, t)]$$

In one transverse dimension, for instance, one obtains

$$[P_L(x, z=0, t)] = \sum_{k=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{4\sigma(\pi k)^2}{D} \frac{t}{L^3} \cos\left(\frac{2\pi k x}{L}\right) \times \exp \frac{-a_n (4D)^{-1/3} \sigma^{2/3} (2\pi k)^{4/3} t}{L^{4/3}} \tag{6.4}$$

In the long-time limit it goes to a uniform front, but the longest transient corresponds to the term $k=1$, $n=1$ in this series. In particular, the long-time limit for the *integrated diffusion front* can be obtained as (keeping only the modes $k=0, 1$)

$$\int dz\,[P_L(x,z,t)] = \frac{1}{L} + \text{const}\cdot\cos\left(\frac{2\pi x}{L}\right) \times \exp\left\{-1.74..(4D)^{-1/3}\sigma^{2/3}\left(\frac{2\pi}{L}\right)^{4/3} t\right\} + \text{h.o.t.} \qquad (6.5)$$

which should easily be checked by numerical simulation.

The physical origin of the correspondence is the following. Slow relaxation in the x direction (or compressed walks) corresponds to *stretched*, self-repulsive walks in the z direction. The strength of the self-repulsion in simply controlled by the radius L of the cylinder.

In higher transverse dimension one would obtain, for instance, a decay of the slowest transient $k=1$ proportional to $t^{\gamma-1}\exp(-s_c t)$; one can even observe in principle the full diffusion front of the polymer.

Note that the discussion of the universality of the tails of the MdM model corresponding to slow diffusion along x follows closely the one of Section 4 on the spin depolarization problem. Certainly (4.1)–(4.3) hold in the scaling regime $t \sim L^{4/3}$. There might be, however, nonuniversal tails: consider, for instance, a distribution of velocities $V(z)$ with a delta function weight at $V=0$. Clearly a Lifschitz argument shows that with a probability $\exp(-t^{d/(d+2)})$ a tracer trapped inside a region of $V=0$ will not diffuse at all along x. These tails will ultimately correct the result (4.3).

7. CONCLUSION

In conclusion, we have described some connections between the Matheron–de Marsilly model, the depolarization of spins in a random field, diffusion in the presence of sources and sinks, and the well-studied models of the Edwards chain and the electron in a random potential. We have shown that these connections are useful. In particular, we have obtained analytically the diffusion front in the MdM model: there are very few examples of diffusion in random flows where the front is nontrivial and can be computed exactly. We have obtained exactly the leading decay of the total magnetization for one-dimensional diffusion in a random field, which can be measured in NMR experiments.

Since this work was completed a number of papers on closely related subjects have come to my attention, most of them kindly indicated to me by the referees. First, as was mentioned in the Introduction, probabilists have studied in great detail the relation between the Edwards chain and random Schrödinger operators. There is now a precise formulation by probabilists of the theories of Thouless (and of Balian–Toulouse) without the use of the $n=0$ trick. The interested reader can find a detailed analysis along these lines of the Edwards partition function in the review by Westwater.[31] In a recent study, March and Sznitman[32] generalize the theorems to arbitrary potentials (i.e., more general than ϕ^4) and study the partition function in the t domain. Second, Kesten and Spitzer[33] obtained a proof of the universality of the diffusion front in MdM-like models. They did not, however, to the best of my understanding, obtain an exact expression for the front. Finally, there are very rencent related works by Zumofen *et al.*[34] and Avelameda and Majda.[35]

APPENDIX

In this Appendix we write the general expression for the moments of the horizontal displacement in the DMM model. It is $x(t)=\int_0^t d\tau\, V(z(\tau))$ for a given realization of the random potential $V(z)$ and a given thermal history $z(\tau)$ of the vertical coordinate. We denote by $\langle\cdots\rangle_z^{(\mathrm{un})}$ the unnormalized thermal average over vertical paths such that $z(t)=z$ [we take always $z(0)=0$]. The usual thermal average $\langle\cdots\rangle$ without condition on the endpoint is thus $\langle\cdots\rangle=\int dz\,\langle\cdots\rangle_z^{(\mathrm{un})}$. One has

$$\begin{aligned}\langle x^n(t)\rangle_z^{(\mathrm{un})}=n!\int_0^t d\tau_n\int_0^{\tau_n} d\tau_{n-1}\cdots\int_0^{\tau_2} d\tau_1\\ \times\int dr_1\cdots dr_n\,\{P_0(r_1,\tau_1)\,P_0(r_2-r_1,\tau_2-\tau_1)\\ \times\cdots P_0(r_n-r_{n-1},\tau_n-\tau_{n-1})\,P_0(z-r_n,t-\tau_n)\,V(r_1)\cdots V(r_n)\}\end{aligned}$$

where $P_0(z'-z,\ t'-t)$ is the propagator of the diffusion along the vertical direction (we have supposed translational invariance in t and z). The corresponding expression for $\langle x^n(t)\rangle$ is identical with the term $P_0(z-r_n,\ t-\tau_n)$ deleted. After a Laplace transformation, this can be rewritten

$$\begin{aligned}\langle x^n(s)\rangle_z^{(\mathrm{un})}=n!\int dr_1\cdots dr_n\,V(r_1)\cdots V(r_n)\,P_0(r_1,s)\,P_0(r_2-r_1,s)\\ \times\cdots P_0(r_n-r_{n-1},s)\,P_0(z-r_n,s)\end{aligned}$$

Thus, for the generating function of these moments one obtains

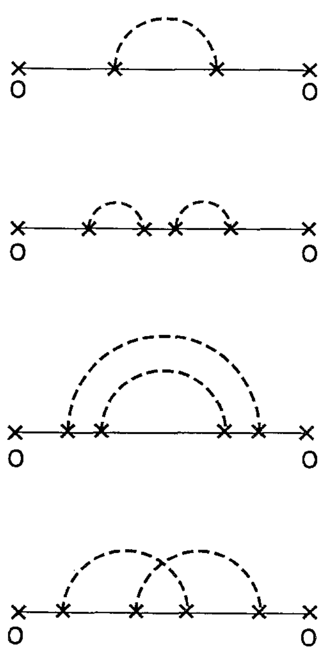
$$\int ds\,e^{-st}\int dx\,e^{-i\lambda x}P(x,z,t)=\sum_{n=0}^{\infty}(-i\lambda)^n\langle z|\,G_0^{-1}VG_0^{-1}V\cdots G_0^{-1}|0\rangle$$

with n factors of V on the r.h.s. We have defined the pure diffusion operator $\langle z| G_0^{-1} |z'\rangle = P_0(z'-z, s)$. For pure continuum diffusion, $G_0 = s - D\nabla^2$. The formal resummation of the above formula thus gives

$$\int ds\, e^{-st} \int dx\, e^{-\lambda x} P(x, z, t) = \langle z| \frac{1}{G_0 + \lambda V} |0\rangle$$

and provides a demonstration of the Feyman–Kac formula (2.5) which is valid (formally) for any configuration $V(z)$.

The unaveraged above formula can be represented very simply by the diagram in Fig. 3a, where momentum is conserved at the vertex and all moments are integrated on. There is a factor $G_0^{-1}(q, s)$ per solid propagator. Let us now consider Gaussian disorder with $V(q) = \int dz\, e^{iqz} [V(0)\, V(z)]$. The averages $[\langle x^n(s)\rangle_0^{(\mathrm{un})}]$ are represented by similar diagrams where disorder lines are paired up in all possible ways with a factor $V(q)$ for each disorder line of momentum q. Represented in Fig. 3 are the single diagram contributing to the mean square displacement $n = 2$ and the three diagrams for $n = 4$. For an arbitrary discrete model the only property of the propagator $G_0^{-1}(q, s)$ which is needed is that for small



(a)

(b)

Fig. 3. (a) Graphs contributing to the mean square displacement. The dashed lines are disorder propagators. (b) Graphs contributing to the fourth moment of the displacement.

s it takes the form $G_0(q, s) = s + Dq^2 + \text{h.o.t.}$, which defines the diffusion coefficient D. The normalization gives $\int (dq/2\pi)\, G_0^{-1}(q, s) \sim (4Ds)^{-1/2}$.

The diagram for $n = 2$ is

$$[\langle x^2(s)\rangle_0^{(\text{un})}] \sim 2! \int \frac{dq_1\, dq_2}{(2\pi)^2} \frac{V(q_1 - q_2)}{Dq_2^2 + s} \left(\frac{1}{Dq_1^2 + s}\right)^2 \sim \frac{V(0)}{4Ds^2}$$

It is easy to check that only $\sigma = V(0) = \int dz\, [V(0)\, V(z)]$ and D, when they exist, contribute to the dominant divergence when $s \to 0$. These dominant infrared divergence do not depend on an ultraviolet cutoff (or microscopic details). This property remains true for all moments. In order to check the predictions of the exact solution in Section 3.5 we have also calculated the moment $n = 4$. Diagrams in 3b give $4!\, D^{-3/2} s^{-7/2}$ multiplied by 3/64, 1/32, and 5/256, respectively, for a total

$$[\langle x^4(s)\rangle_0^{(\text{un})}] = 4! \left(\frac{5}{16}\right)^2 D^{-3/2} s^{-7/2}$$

We inverse-Laplace-transformed and divided by the normalization (probability density at layer $z = 0$) to find the result displayed in Section 3.5.

A similar calculation can be performed for the unrestricted thermal average (the external momentum is set equal to zero and not integrated upon). Thus

$$[\langle x^2(s)\rangle] = \frac{2}{s^2} \int \frac{dq}{2\pi} \frac{V(q)}{Dq^2 + s} \sim \frac{1}{D^{1/2} s^{5/2}}$$

and thus

$$[\langle x^2(t)\rangle] \sim \frac{4}{3(\pi D)^{1/2}}\, t^{3/2}$$

Note, by comparing with the above result, that walks which come back to the same layer $z = 0$ have a horizontal diffusion enhanced by a factor of roughly 4 compared to the average walk. This is, of course, because they visit more often (roughly twice) the same layer (see Section 1).

Reasonable deviations from a Gaussian are unimportant in the long-time limit. For instance, consider the diagram for the $n = 4$ moment corresponding to a fourth-order connected cumulant $[VVVV]_c$. One obtains

$$[VVVV]_c(q = 0) \int \frac{dq_0}{2\pi(Dq_0^2 + s)^2} \left(\int \frac{dq}{2\pi(Dq^2 + s)}\right)^3 \sim s^{-3}$$

which is thus subdominant compared to the above divergence $s^{-7/2}$ coming from the contribution proportional to σ, provided $[VVVV]_c(q = 0)$ exists.

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